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Alternative LMI Characterizations for Fractional-order Linear Systems

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Abstract: This paper focuses on the linear matrix inequality (LMI) characterizations of fractional-order linear systems. Based on the generalized Kalman-Yakubovic-Popov (KYP) lemma, two bounded real lemmas of fractional-order linear systems are introduced with respect to two different norms respectively. Then an new bounded real lemma is proposed with more degrees of freedom. In terms of a set of LMIs then, it is generalized for a class of fractional-order uncertain linear systems with the convex polytopic uncertainties , which forms less conservative constraints on $\mathcal{H}_\infty$ performance. Finally this result is demonstrated in a numerical example.

Key Words: Fractional-order System, Bounded Real Lemma, Linear Matrix Inequality, Polytopic Uncertainty

1 Introduction

In recent years, an extensive research has been devoted to extend linear matrix inequality (LMI) characterizations for stability and performance of fractional-order linear systems (FOLS). Starting from the work of [1–3], LMI have gained increasing interest in FOLS and its control theory. The LMI stability conditions for FOLS were investigated in [3–5] and several LMI-based stability criterions were verified. Due to the potentialities of FOLS to improve classical systems stability, there are many works on robust stability and stabilization of FOLS [6–15, 17–19, 23, 24] and the references therein.

LMI-based robust stabilization is fundamental to the fractional-order uncertain systems control. As a way of efficiently solving the robust stability and stabilization problem, LMI-based approach was presented in [6–8] that provided the sufficient conditions and designing methods of state feedback controllers for fractional-order interval systems. [9–11] extended the LMI-based robust stability to the fractional-order interval systems with linear couplings among the fractional order, the system matrix and the input matrix. The static output feedback controller and observer-based controller were derived in [12, 13] by using matrix's singular value decomposition and LMI techniques. In [14], the robust decentralized control of perturbed fractional-order linear interconnected systems with structured and unstructured perturbations was derived. The maximal robust stabilizability perturbation bound was computed by LMI solvers for the fractional-order uncertain systems and the corresponding linear state feedback stabilizing controller was given in [15]. Although the state feedback $\mathcal{H}_\infty$ control and the static output feedback $\mathcal{H}_\infty$ control of commensurate FOLS were firstly presented in [18] and [19] respectively, the LMI-based $\mathcal{H}_\infty$ control and $\mathcal{H}_\infty$ performances of FOLS remain some works to be done, which is an important study of FOLS that we intend to motivate.

To authors' best knowledge, the concept of $\mathcal{H}_\infty$-norm for FOLS was investigated in [1, 2] early and in [16] recently. It was shown in [17] that the standard Nehari problem for FOLS can be solved by the generalized Youla parameterization. By use of the bounded real lemma for FOLS, the $\mathcal{H}_\infty$ model reduction of FOLS was proposed in [20] and the $\mathcal{H}_\infty$ performance was characterized in [21]. The key idea of evaluating the $\mathcal{H}_\infty$-norm of FOLS using LMI-based approach is the bounded real lemma [1, 18, 20, 21]. As a special case of the generalized Kalman-Yakubovic-Popov (KYP) lemma, the bounded real lemmas for FOLS still need to be extended [21, 22].

In our contributions, based on the generalized KYP lemma, two bounded real lemmas of FOLS are introduced with respect to two different norms respectively. To generalize the classical results in [23] to one of fractional-order uncertain systems, i.e., fractional-order polytopic systems (FOPS) [24, 25], an alternative bounded real lemma is proved with more degrees of freedom. Then it is generalized with the convex polytopic uncertainties in terms of a set of LMIs. The new bounded real lemma for FOPS provides an alternative approach to deal with the system $\mathcal{H}_\infty$ performance with less conservative. Finally, this result is demonstrated in a numerical example.

We use the following notations. The transpose of a matrix A is denoted by A^T. $\bar{A}$,A^* denote the conjugate and the conjugate transpose of a matrix A respectively. I_n denotes the identity matrix of dimension n. The symbol $\mathbb{H}_n$ stands for the set of $n \times n$ Hermitian matrices. $\mathbb{R}, \mathbb{C}$ are the sets of real and complex numbers respectively. $A + A^*$ is denoted by $Sym\{A\}$ and $\bar{\sigma}(A)$ is the maximum singular value of a matrix A. $\otimes$ is the Kroneckers product. For $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{H}_{n+m}$, a function $\sigma : \mathbb{C}^{n\times m} \times \mathbb{H}_{n+m} \to \mathbb{H}_m$ is defined by

$$\sigma(A, B) = \begin{bmatrix} A \\ I_m \end{bmatrix}^* B \begin{bmatrix} A \\ I_m \end{bmatrix}.$$

2 Preliminaries

In this paper, as the physical meaning is concerned, we use the Caputo fractional-order derivative.

Definition 1. *[4] Let $f(t)$ is a real continuously differentiable function. The Caputo fractional-order derivative with*

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order $0 < \nu < 1$ *on* $t > 0$ *is defined by*

$$D_t^{\nu} = \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{(\nu-n+1)}} d\tau,$$

where $n = \lceil \nu \rceil, \nu > 0$, $\lceil \cdot \rceil$ *is the ceiling function.*

The commensurate fractional-order linear time invariant system can be described by the following pseudo-state space representation (A, B, C, D, ν).

$$\begin{cases} D^{\nu}x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the pseudo-state, $u(t) \in \mathbb{R}^m$ is the input signal, $y(t) \in \mathbb{R}^p$ is the output. D^{ν} is the Caputo fractional-order differentiation operator with fractional order ν. A, B, C, D are constant matrices with appropriate sizes. The transfer function between $u(t)$ and $y(t)$ is $G(s) = C(s^{\nu}I - A)^{-1}B + D$ and its impulse response is $g(t) = \mathcal{L}^{-1}\{G(s)\}$.

We only consider the case $0 < \nu < 1$ for simplicity, because it always possible by including the additional pseudo-states. The minimal realization is very useful for obtaining pseudo-state space representations from fractional-order transfer functions.

Assumption 1. *The fractional-order linear time invariant system* (A, B, C, D) *is the minimal realization of order* ν *if, and only if, it is concurrently observable and controllable.*

Remark 1. *As the heredity and nonlocality of fractional-order differentiation operators, the* $x(t)$ *does not reflect the true dynamics and therefore are named "pseudo-state" or "quasi-state". The initialization problem plays an important role in real applications of FOLS.*

Definition 2. *[16] A fractional-order linear time invariant system is bounded-input bounded-output (BIBO) stable if, and only if,* $y(t) = g(t) \star u(t) \in \mathcal{L}_{\infty}(\mathbb{R}^+, \mathbb{R}^p)$ *for any input* $u(t) \in \mathcal{L}_{\infty}(\mathbb{R}^+, \mathbb{R}^m)$*, where* $\star$ *is the convolution operator.*

Theorem 1. *[16] A fracitonal-order linear time invariant system* (A, B, C, D, ν) *with* $0 < \nu < 1$ *is BIBO stable if, and only if,* $|arg(spec(A))| > \nu\frac{\pi}{2}$*, where* $spec(A)$ *is the spectrum of all eigenvalues of* A*.*

Theorem 2. *[5] A fractional-order linear time invariant system* (A, B, C, D, ν) *with* $0 < \nu < 1$ *is BIBO stable if, and only if, there exists a positive definite matrix* $X = X^* \in \mathbb{H}_n$ *such that*

$$(rX + \bar{r}\bar{X})^T A^T + A(rX + \bar{r}\bar{X}) < 0,$$

where $r = e^{j\theta}, \theta = (1-\nu)\frac{\pi}{2}$*.*

Definition 3. *[2, 16]* $\mathcal{L}_{\infty}$*-norm of a system* $G(s)$ *in* $\mathcal{L}_{\infty}$ *space is denoted by* $\|G(s)\|_{\mathcal{L}_{\infty}} = sup_{\omega \in \mathbb{R}}\bar{\sigma}(G(j\omega))$*.* $\mathcal{H}_{\infty}$*-norm of a system* $G(s)$ *in* $\mathcal{H}_{\infty}$ *space is denoted by* $\|G(s)\|_{\mathcal{H}_{\infty}} = sup_{Re(s)>0}\bar{\sigma}(G(s))$*.*

For the proper and stable $G(s)$, the two norms are equivalent. The frequency ranges can be characterized in the generalized Kalman-Yakubovic-Popov (KYP) lemma as the set of complex numbers that represent a certain class of curves [22].

$$\Lambda(\Phi, \Psi) = \{\lambda \in \mathbb{C} | \sigma(\lambda, \Phi) = 0, \sigma(\lambda, \Psi) \geq 0\}, \tag{2}$$

where $\Phi, \Psi \in \mathbb{H}_2$.

Define $\bar{\Lambda} = \Lambda \cup \{\infty\}$ if Λ is bounded, otherwise $\bar{\Lambda} = \Lambda$.

Theorem 3. *(Generalized KYP [22]) Let matrcies* $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \Theta \in \mathbb{H}_{n+m}$*,* $F = \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix} \in \mathbb{C}^{2n \times (m+n)}$*, and* $\Phi, \Psi \in \mathbb{H}_2$*. Define* Λ *and* $\bar{\Lambda}$ *as (2). Suppose that* Λ *represents curves on the complex plane. Denote* N_{λ} *the null space of* $\Gamma_{\lambda}F$*,where*

$$F_{\lambda} = \begin{cases} [I_n - \lambda I_n], \lambda \in \mathbb{C} \\ [0 - I_n], \lambda = \infty \end{cases}.$$

The following statements are equivalent.

(i) $N_{\lambda}^*\Theta N_{\lambda} < 0, \forall \lambda \in \bar{\Lambda}(\Phi, \Psi)$*;*

(ii) There exists $P, Q \in \mathbb{H}_n$ *such that* $Q > 0$ *and* $F^*(\Phi \otimes P + \Psi \otimes Q)F + \Theta < 0$*.*

Moreover, if $rank(\Gamma_{\lambda}F) = n, \forall \lambda \in \mathbb{C} \cup \{\infty\}$ *is satisfied. The following statements are equivalent.*

(i) $N_{\lambda}^*\Theta N_{\lambda} \leq 0, \forall \lambda \in \bar{\Lambda}(\Phi, \Psi)$*;*

(ii) There exists $P, Q \in \mathbb{H}_n$ *such that* $Q \geq 0$ *and* $F^*(\Phi \otimes P + \Psi \otimes Q)F + \Theta \leq 0$*.*

When Θ, F, Φ, Ψ are all real matrices, P, Q in (ii) can be restricted to be real without loss of generality. It is obvious that N_{λ}, the null space of $\Gamma_{\lambda}F$ can be specified by

$$N_{\lambda} = \begin{bmatrix} H(\lambda) \\ I_m \end{bmatrix}, H(\lambda) = (\lambda I - A)^{-1}B.$$

To extend the KYP lemma for FOLS, the curves described by Λ can be relaxed to Υ as the following corollary.

Corollary 1. *[21] If the set* $\Lambda(\Phi, \Psi)$ *is replaced by* $\Upsilon(\Phi, \Psi) = \{\lambda \in \mathbb{C} | \sigma(\lambda, \Phi) \geq 0, \sigma(\lambda, \Psi) \geq 0\}$*, the condition* $N_{\lambda}^*\Theta N_{\lambda} < 0$ *holds for any* $\lambda \in \Upsilon$ *if there exists positive definite matrices* $P, Q \in \mathbb{H}_n$ *such that* $F^*(\Phi \otimes P + \Psi \otimes Q)F + \Theta < 0$*.*

3 Bounded Real Lemmas for FOLS

In this section, based on the generalized KYP, two bounded real lemmas of FOLS are introduced in LMI characterizations. It should be noted these results are not new [21], alternatively, they provide another efficient approaches of computing the $\mathcal{H}_{\infty}$-norm or the $\mathcal{L}_{\infty}$-norm of FOLS.

The frequency range of FOLS with order $0 < \nu < 1$ can be characterized as a curve (or curves) on the complex plane. According to (2), define $\theta = (1-\nu)\frac{\pi}{2}$, the curve of the frequency constraint on $\|G(s)\|_{\mathcal{L}_{\infty}}$ is

$$\Lambda(\begin{bmatrix} 0 & e^{-j\theta} \\ e^{j\theta} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1-\nu \\ 1-\nu & 0 \end{bmatrix}). \tag{3}$$

The curve of the frequency constraint on $\|G(s)\|_{\mathcal{H}_{\infty}}$ is $\{\lambda \in \mathbb{C} | \lambda = (s)^{\nu}, Re(s) \geq 0\}$, which can characterized by

$$\Upsilon(\begin{bmatrix} 0 & e^{j\theta} \\ e^{-j\theta} & 0 \end{bmatrix}, \begin{bmatrix} 0 & e^{-j\theta} \\ e^{j\theta} & 0 \end{bmatrix}). \tag{4}$$

Remark 2. *Although the stability of FOLS with order* $0 < \nu < 1$ *is not convex set in the complex plane [3, 4], the frequency range for computing* $\mathcal{H}_{\infty}$ *norm is convex. Therefore, it can be characterized by the quadratic function.*

The upper triangular elements in Hermitian matrices are denoted by the symbol $*$ briefly.

Theorem 4. *Let the transfer function $G(s) = C(s^\nu I - A)^{-1}B + D$ of (1) and a positive real number $\gamma > 0$.Then $\|G(s)\|_{\mathcal{L}_\infty} < \gamma$ holds if, and only if, there exists matrices $P, Q \in \mathbb{H}_n$ such that $Q > 0$ and*

$$\begin{bmatrix} Sym\{(e^{j\theta} + (1-\nu)Q)A\} & * & * \\ B^T(e^{-j\theta} + (1-\nu)Q) & -\gamma I & * \\ C & D & -\gamma I \end{bmatrix} < 0. \quad (5)$$

Proof. According to Definition 3 and $\bar{\sigma}(G(s)) = \bar{\sigma}(G^*(s)) = \bar{\sigma}(G(s^*))$ holds for any $s \in \mathbb{C}$, we have

$$sup_{\omega\geq 0}\bar{\sigma}(G(j\omega)) = sup_{\omega\geq 0}\bar{\sigma}(G(-j\omega)) = sup_{\omega\leq 0}\bar{\sigma}(G(j\omega)),$$

$$sup_{\omega\geq 0}\bar{\sigma}(G(j\omega)) = sup_{\omega\in\mathbb{R}}\bar{\sigma}(G(-j\omega)) = \|G(s)\|_{\mathcal{L}_\infty}.$$

Therefore, $\|G(s)\|_{\mathcal{L}_\infty} = sup_{\omega\geq 0}\{C(\lambda(\omega)I - A)^{-1}B + D\} < \gamma$ and can be reduced into $G(j\omega)^*G(j\omega) < \gamma^2 I_m$.

The frequency curves $\{\lambda \in \mathbb{C}|\lambda(\omega) = e^{j\nu\frac{\pi}{2}}\omega^\nu, \omega > 0\}$ can be represented by (3), i.e.,

$$\Lambda\Phi, \Psi), \Phi = \begin{bmatrix} 0 & e^{-j\theta} \\ e^{j\theta} & 0 \end{bmatrix}, \Psi = \begin{bmatrix} 0 & 1-\nu \\ 1-\nu & 0 \end{bmatrix}.$$

Let $H(\lambda) = (\lambda I - A)^{-1}B, \Theta = \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - \gamma^2 I_m \end{bmatrix}, N_\lambda = \begin{bmatrix} H(\lambda) \\ I_m \end{bmatrix}$, we have $N_\lambda^*\Theta N_\lambda < 0$.

According to Theorem 3, we have

$$\begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix}^T (\begin{bmatrix} 0 & e^{-j\theta} \\ e^{j\theta} & 0 \end{bmatrix} \otimes P + \begin{bmatrix} 0 & 1-\nu \\ 1-\nu & 0 \end{bmatrix} \otimes Q) \times$$

$$\begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix} + \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - \gamma^2 I_m \end{bmatrix} < 0.$$

That is

$$\begin{bmatrix} sym\{(e^{j\theta}P + (1-\nu)Q)A\} & * \\ B^T(e^{-j\theta}P + (1-\nu)Q) & -\gamma^2 I_m \end{bmatrix} + [C\ D]^T[C\ D] < 0.$$

Then using complex Shur complement [18], and multiplying with $diag[\gamma^{-\frac{1}{2}}I_n, \gamma^{-\frac{1}{2}}I_m, \gamma^{\frac{1}{2}}I_p]$ on both left and right sides, the final condition (5) is obtained. □

Theorem 5. *Let the transfer function $G(s) = C(s^\nu I - A)^{-1}B + D$ of (1), and a positive real number $\gamma > 0$. Then $\|G(s)\|_{\mathcal{H}_\infty} < \gamma$ holds if there exist positive definite matrices $P, Q \in \mathbb{H}_n$ such that*

$$\Xi_1 = \begin{bmatrix} Sym\{(e^{-j\theta}P + e^{j\theta}Q)A\} & * & * \\ B^T(e^{j\theta}P + e^{-j\theta}Q) & -\gamma I & * \\ C & D & -\gamma I \end{bmatrix} < 0. \quad (6)$$

Proof. Note Definition 3, and $\|G(s)\|_{\mathcal{H}_\infty} < \gamma$ can be reduced into $G(s)^*G(s) < \gamma^2 I_m$.

The curve $\{\lambda \in \mathbb{C}|\lambda = (s)^\nu, Re(s) > 0\}$ can be characterized by (4),i.e.,

$$\Upsilon\Phi, \Psi), \Phi = \begin{bmatrix} 0 & e^{j\theta} \\ e^{-j\theta} & 0 \end{bmatrix}, \Psi = \begin{bmatrix} 0 & e^{-j\theta} \\ e^{j\theta} & 0 \end{bmatrix}.$$

Let $H(\lambda) = (\lambda I - A)^{-1}B, \Theta = \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - \gamma^2 I_m \end{bmatrix}, N_\lambda = \begin{bmatrix} H(\lambda) \\ I_m \end{bmatrix}$, we have $N_\lambda^*\Theta N_\lambda < 0$.

According to Theorem 3, we have

$$\begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix}^T (\begin{bmatrix} 0 & e^{j\theta} \\ e^{-j\theta} & 0 \end{bmatrix} \otimes P + \begin{bmatrix} 0 & e^{-j\theta} \\ e^{j\theta} & 0 \end{bmatrix} \otimes Q) \times$$

$$\begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix} + \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - \gamma^2 I_m \end{bmatrix} < 0.$$

That is

$$\begin{bmatrix} sym\{(e^{-j\theta}P + e^{j\theta}Q)A\} & * \\ B^T(e^{j\theta}P + e^{-j\theta}Q) & -\gamma^2 I_m \end{bmatrix} + [C\ D]^T[C\ D] < 0.$$

Then using complex Shur complement [18], and multiplying with $diag[\gamma^{-\frac{1}{2}}I_n, \gamma^{-\frac{1}{2}}I_m, \gamma^{\frac{1}{2}}I_p]$ on both left and right side, the final condition is obtained similarly. □

Remark 3. *Although $\mathcal{L}_\infty$-norm bounded real lemma is a sufficient and necessary condition, the systems stability cannot be guaranteed and it is difficult to design stabilization controllers. The $\mathcal{H}_\infty$-norm can be obtained by (5) if the G(s) is stable, which is less conservative than (6).*

The $\mathcal{H}_\infty$-norm of FOPS can be computed from $\mathcal{H}_\infty$-norm bounded real lemma directly, the state feedback $\mathcal{H}_\infty$ control problem were proposed in [18].

4 Bounded Real Lemma for FOPS

Since polytopic domains are a class of quite general representations of parameter uncertainties, fractional-order systems with real convex bounded uncertainties have received considerable attention [23, 24] recently. The real convex uncertainties can be described as follows.

Definition 4. *The parametric uncertainties $(A, B, C, D)(a)$ are affine functions of the uncertain parametric vector $a \in \mathbb{R}^N$ described by the convex pplytope $\mathcal{M}$ with vertices at $(A_j, B_j, C_j, D_j), j = 1, \ldots, N$.*

$$\mathcal{M} = \left\{ \begin{array}{l} (A, B, C, D)(a) = \sum_{j=1}^N a_j(A_j, B_j, C_j, D_j) \\ |\sum_{j=1}^N a_j = 1, a_j \geq 0, j = 1, \ldots, N. \end{array} \right\}.$$

A fractional-order polytopic system (FOPS) can be described by the uncertain pseudo-state space representation $(A(a), B(a), C(a), D(a))$.

$$\left\{ \begin{array}{l} D^\nu x(t) = A(a)x(t) + B(a)u(t) \\ y(t) = C(a)x(t) + D(a)u(t) \end{array} \right. . \quad (7)$$

where $x(t) \in \mathbb{R}^n$ is the pseudo-state, $u(t) \in \mathbb{R}^m$ is the input signal, $y(t) \in \mathbb{R}^p$ is the output. D^ν is the fractional-order differentiation operator with order ν. $M(a) = (A(a), B(a), C(a), D(a))$ are uncertain matrices in $\mathcal{M}$ with appropriate sizes. The transfer function between $u(t)$ and $y(t)$ is $G(s) = C(a)(s^\nu I - A(a))^{-1}B(a) + D(a)$.

Now, Theorem 5 can be generalized directly for fractional-order uncertain systems with the convex polytopic uncertainties in terms of a set of LMIs.

Theorem 6. *Let the transfer function $G(s) = C(a)(s^\nu I - A(a))^{-1}B(a) + D(a)$ of (8), and a real number $\gamma > 0$. Then $\|G(s)\|_{\mathcal{H}_\infty} < \gamma$ holds if there exist positive definite matrices $P, Q \in \mathbb{H}_n$ such that*

$$\Xi_j = \begin{bmatrix} Sym\{(e^{-j\theta}P + e^{j\theta}Q)A_j\} & * & * \\ B_j^T(e^{j\theta}P + e^{-j\theta}Q) & -\gamma I & * \\ C_j & D_j & -\gamma I \end{bmatrix} < 0,$$

for all $j = 1, \ldots, N$.

Proof. Consider $\sum_{j=1}^N a_j \Xi_j =$

$$\begin{bmatrix} Sym\{(e^{-j\theta}P + e^{j\theta}Q)A(a)\} & * & * \\ B(a)^T(e^{j\theta}P + e^{-j\theta}Q) & -\gamma I & * \\ C(a) & D(a) & -\gamma I \end{bmatrix} < 0.$$

By use of Theorem 5, the sufficiency can be demonstrated. □

On the other hand, an alternative bounded real lemma for FOLS is introduced based on Theorem 5, which is motivated by the results in [23]. Then it can be generalized for the convex polytopic uncertainties in terms of a set of LMIs.

Theorem 7. *Let the transfer function $G(s) = C(s^\nu I - A)^{-1}B + D$ of (1), and a positive real number $\gamma > 0$. Then $\|G(s)\|_{\mathcal{H}_\infty} < \gamma$ holds if there exist positive definite matrices $P, Q \in \mathbb{H}_n$ and $\Pi, \Sigma, E_1, F_1 \in \mathbb{H}_n, E_i, F_i, i = 2, 3$ such that (8) is satisfied.*

Proof. The equivalence of the LMI representations in Theorem 5 and Theorem 6 can be established as followings.

($\Xi_1' \Rightarrow \Xi_1$) Suppose that there exist positive definite $P, Q \in \mathbb{H}_n, \Pi, \Sigma$ and $E_i, F_i, i = 1, 2, 3$ such that the equivalent LMI representation holds.

Choose the matrix transform T with full row rank as

$$T = \begin{bmatrix} I & 0 & 0 & A^T & -A^T \\ 0 & I & 0 & B^T & -B^T \\ 0 & 0 & I & 0 & 0 \end{bmatrix}.$$

It is easy to show that $T\Xi_1'T^T = \Xi_1$. Then using Theorem 5, we have $\|G(s)\|_{\mathcal{H}_\infty} < \gamma$.

($\Xi_1 \Rightarrow \Xi_1'$) According to Theorem 5, there exist positive definite $P, Q \in \mathbb{H}_n$ such that $\Xi_1 < 0$. This condition can be reduced to that there exists a scalar $\epsilon > 0$ such that

$$\begin{bmatrix} Sym\{(e^{-j\theta}P + e^{j\theta}Q)A\} + \epsilon A^T A & * & * \\ B^T(e^{j\theta}P + e^{-j\theta}Q) + \epsilon B^T A & \epsilon B^T B - \gamma I & * \\ C & D & -\gamma I \end{bmatrix}$$
< 0.

By use of the complex Schur complement [18], we have

$$\begin{bmatrix} Sym\{(e^{-j\theta}P + e^{j\theta}Q)A\} & * & * & * & * \\ B^T(e^{j\theta}P + e^{-j\theta}Q) & -\gamma I & * & * & * \\ C & D & -\gamma I & * & * \\ \epsilon A & \epsilon B & 0 & -2\epsilon I & * \\ -\epsilon A & -\epsilon B & 0 & 0 & -2\epsilon I \end{bmatrix}$$
< 0.

Set $\Pi = \Sigma = \epsilon I, E_1 = F_1 = -e^{-j\theta}P - e^{j\theta}Q, E_i = F_i = 0, i = 2, 3$, it is easy to see that Ξ_1' holds.

So far, the equivalence of the LMI representations of $\Xi_1 < 0$ and $\Xi_1' < 0$ is proved completely. □

In Theorem 6, one could solve the $\mathcal{H}_\infty$-norm problem by seeking a single pair P, Q for all LMIs associated with the vertices of the polytope. On the other have, using Theorem 7, an alternative bounded real lemma for FOPS can be developed here, which reduces the conservation of Theorem 6.

Theorem 8. *Let the transfer function $G(s) = C(a)(s^\nu I - A(a))^{-1}B(a) + D(a)$ of (8), and a positive real number $\gamma > 0$. Then $\|G(s)\|_\infty < \gamma$ holds if there exist positive definite matrices $P_j, Q_j \in \mathbb{H}_n$ and $\Pi, \Sigma, E_1, F_{1j} \in \mathbb{H}_n, j = 1, \ldots, N; F_{ij}, E_i, i = 2, 3; j = 1, \ldots, N$ such that (9) is satisfied for all $j = 1, \ldots, N$.*

Proof. Consider Theorem 6, let $P(a) = \sum_{j=1}^N a_j P_j, Q(a) = \sum_{j=1}^N a_j Q_j, F_i(a) = \sum_{j=1}^N a_j F_{ij}, i = 1, 2, 3$.

Thus we have $\sum_{j=1}^N a_j \Xi_{1j}' = (10)$.

By use of Theorem 6, the sufficiency can be demonstrated directly. □

5 Numerical computation

In this section, a numerical example is presented to demonstrate the difference between Theorem 6 and Theorem 8. The $\mathcal{H}_\infty$-norm computation of FOPS can be transferred into semidefinite programming(SDP) problems. To solve them we use CVX, a package for specifying and solving convex programs [26, 27].

Exampe 1. *Consider an uncertain FOLS (1) with $\nu = 0.5$,*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \rho \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & -2 \end{bmatrix}, D = 0.$$

where the uncertain interval is $|\rho + 6| \leq 3$. The system is stable and we need to compute the system $\mathcal{H}_\infty$-norm.

It is obvious that this system can be represented by FOPS (7) with $N = 2$,

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -9 \end{bmatrix},$$
$$B_1 = B_2 = B, C_1 = C_2 = C, D_1 = D_2 = 0.$$

The $\mathcal{H}_\infty$-norm computation can be formed as the following SDPs. By use of Theorem 6, we have

$$\min \quad \gamma \tag{11}$$
$$\begin{aligned} s.t. \quad & \Xi_1 < 0, \\ & \Xi_2 < 0, \\ & P, Q > 0, P, Q \in \mathbb{H}_2. \end{aligned} \tag{12}$$

By solving this problem, the system $\mathcal{H}_\infty$-norm is $\gamma = 4.52$.

$$P = \begin{bmatrix} 5.71 & 1.58 + 0.08i \\ 1.58 - 0.08i & 4.55 \end{bmatrix},$$
$$Q = \begin{bmatrix} 8.83\times10^{-12} & 1.20\times10^{-12} - 7.79\times10^{-14} \\ 1.20\times10^{-12} + 7.79\times10^{-14} & 3.09\times10^{-13} \end{bmatrix}.$$

On the other hand, By use of Theorem 8, we have

$$\min \quad \gamma \tag{13}$$
$$\begin{aligned} s.t. \quad & \Xi_{11}' < 0, \\ & \Xi_{12}' < 0, \\ & P_i, Q_i > 0, P_i, Q_i \in \mathbb{H}_2, i = 1, 2, \\ & \Pi, \Sigma, E_1, F_{1i} \in \mathbb{H}_2, i = 1, 2. \end{aligned} \tag{14}$$

$$
\Xi_1^{'} = \begin{bmatrix} -E_1A - A^TE_1^* & * & * & * & * \\ -B^TE_1^* - E_2A & -\gamma I - E_2B - B^TE_2^T & * & * & * \\ e^{-j\theta}P + e^{j\theta}Q + F_1^* + \Sigma^*A & F_2^T + \Sigma^*B & F_3^T & -\Sigma - \Sigma^* & * \\ F_1^* - E_1^* - \Pi^*A & F_2^T - E_2^T - \Pi^*B & F_3^T - E_3^T & 0 & -\Pi - \Pi^* \end{bmatrix} < 0. \quad (8)
$$

$$
\Xi_{1j}^{'} = \begin{bmatrix} -E_1A_j - A_j^TE_1^* & * & * & * & * \\ -B_j^TE_1^* - E_2A_j & -\gamma I - E_2B_j - B_j^TE_2^T & * & * & \\ C_j - E_3A_j & D_j - E_3B_j & -\gamma I & * & * \\ e^{-j\theta}P_j + e^{j\theta}Q_j + F_{1j}^* + \Sigma^*A_j & F_{2j}^T + \Sigma^*B_j & F_{3j}^T & -\Sigma - \Sigma^* & * \\ F_{1j}^* - E_1^* - \Pi^*A_j & F_{2j}^T - E_2^T - \Pi^*B_j & F_{3j}^T - E_3^T & 0 & -\Pi - \Pi^* \end{bmatrix} < 0. \quad (9)
$$

$$
\begin{bmatrix} -E_1A(a) - A(a)^TE_1^* & * & * & * & * \\ -B(a)^TE_1^* - E_2A(a) & -\gamma I - E_2B(a) - B(a)^TE_2^T & * & * & \\ C(a) - E_3A(a) & D(a) - E_3B(a) & -\gamma I & * & * \\ e^{-j\theta}P(a) + e^{j\theta}Q(a) + F_1(a)^* + \Sigma^*A(a) & F_2(a)^T + \Sigma^*B(a) & F_3(a)^T & -\Sigma - \Sigma^* & * \\ F_1(a)^* - E_1^* - \Pi^*A(a) & F_2(a)^T - E_2^T - \Pi^*B(a) & F_3(a)^T - E_3^T & 0 & -\Pi - \Pi^* \end{bmatrix} < 0. \quad (10)
$$

By solving this problem, the system $\mathcal{H}_\infty$-norm is $\gamma = 1.02$.

$$
P_1 = \begin{bmatrix} 3.70 & -2.64 - 3.41i \\ -2.64 + 3.41i & 5.03 \end{bmatrix},
$$

$$
P_2 = \begin{bmatrix} 6.98 & -2.13 - 2.73i \\ -2.13 + 2.73i & 1.72 \end{bmatrix},
$$

$$
Q_1 = \begin{bmatrix} 3.70 & -2.65 + 3.41i \\ -2.65 - 3.41i & 5.03 \end{bmatrix},
$$

$$
Q_1 = \begin{bmatrix} 6.98 & -2.13 + 2.73i \\ -2.13 - 2.73i & 1.72 \end{bmatrix}.
$$

where other variables are omitted here.

It is obvious that the new $\mathcal{H}_\infty$-norm computation (13–14) is more accurate, which is 25% smaller than (11–12). Therefore, the alternative bounded real lemma in Theorem 8 provide a less conservative condition to deal with $\mathcal{H}_\infty$ performance of FOPS than Theorem 6.

Actually we can directly use Theorem 7 to compute the system $\mathcal{H}_\infty$-norm. It can be realized by solving each individual optimization problem (13–14) within the uncertain interval $|\rho + 6| \leq 3$. The system $\mathcal{H}_\infty$-norm can be obtained by the global minimum value.

The relationship between the system $\mathcal{H}_\infty$-norm and the uncertain parameter is shown in Fig. 1.

It is obvious that the global minimum value can be reached at two endpoints and its actual value is about $\gamma = 1.02$.

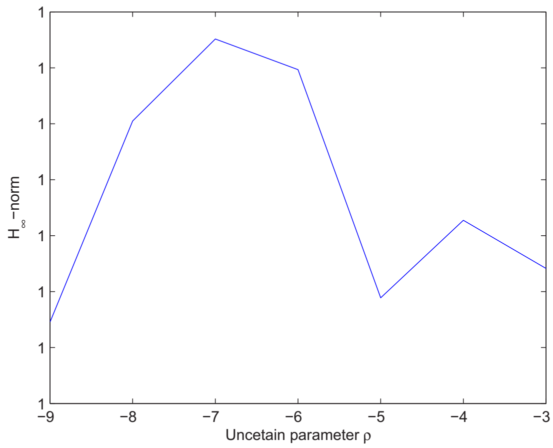

Fig. 1: $\mathcal{H}_\infty$-norm

6 Conclusion and future works

The linear matrix inequality (LMI) characterizations of fractional-order linear systems are surveyed. To motivate LMI-based $\mathcal{H}_\infty$ control of FOLS, two bounded real lemmas of fractional-order linear systems are introduced for two different norms respectively. Besides, an alternative bounded real lemma is proposed and then is generalized for the convex polytopic uncertainties in terms of a set of LMIs.

The future topics can be directed into several aspects. The first will be $\mathcal{H}_\infty$ control of FOLS or FOPS. Besides, some effective bounded real lemmas for FOLS with $\nu > 1$ need to be investigated further.

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